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Spiral Conditions for Splines and Their Applications to Curve Design

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Abstract

Walton & Meek obtained a G^2 fair curve by adding a spiral segment to one end of an existing curve ([1]). The added segments are commonly quadratic, T-cubic, general cubic and PH quintic spirals. We derive the larger regions for the end points of two-parameter general cubic and PH quintic spirals.

1 Introduction

Spirals have several advantages of containing neither inflection points, singularities nor curvature extrema. Such curves are useful in the design of fair curves. Walton & Meek ([1]) considered a G^2 curve design with spiral segments.

The object of this paper is to examine their methods and obtain, in some cases, larger reachable regions for the end points of quadratic, T-cubic, general cubic and PH quintic spiral segments starting from the origin. The added segment passes through the origin and is constrained by its beginning unit tangent vector $(1,0)$ and its beginning curvature. Sections 2-3 treat the cases (i) *starts a non-inflection point* with a radius of curvature, r, and continues with a curvature of increasing magnitude *up to a given non-inflection point*, or (ii): a segment *starts a non-inflection point* with a radius of curvature, r, and continues with a curvature of decreasing magnitude *up to a given non-inflection point.* Figure 1 shows that T-cubic spirals are more flexible than quadratic ones. Sections 4-5 treat the two cases: (iii) a segment *starts an inflection point* with a curvature of increasing magnitude up to a given radius of curvature, r, with a given ending unit tangent vector $(\cos\theta, \sin\theta)$, or (iv) starts a non-inflection point with a given radius of curvature, r, and continues with a curvature of decreasing magnitude *up to an inflection point* with a given ending unit tangent vector $(\cos\theta, \sin\theta)$. Sections 2-5 consider the spiral curve $\boldsymbol{z}(t)(=(x(t), y(t)), 0 \le t \le 1$ and obtain the reachable regions for $(\xi, \eta)(=\boldsymbol{z}(1)/r)$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = \boldsymbol{z}'(t) \times \boldsymbol{z}''(t)/\|\boldsymbol{z}'(t)\|^3 \tag{1.1}$$

where "$\times$" and $\|\bullet\|$ mean the cross product of two vectors and the Euclidean norm, respectively.

2 Quadratic spirals

This section first treats the cases (i) and (ii) for quadratic spirals whose unit tangent vector are not fixed. Require the following conditions for $0 < \theta < \pi/2$:

$$z(0) = (0,0), \quad z'(0) \parallel (1,0), \quad z'(1) \parallel (\cos\theta, \sin\theta), \quad \kappa(0) = 1/r \tag{2.1}$$

to obtain

$$x'(t) = u_0(1-t) + (u_0^2 t/r)\cot\theta, \quad y'(t) = u_0^2 t/r \quad (u_0 > 0) \tag{2.2}$$

Then, with $t = 1/(1+s)$

$$\kappa'(t) = \frac{3r^2(1+s)^4\{rs(r\tan\theta - u_0)\cot\theta + u_0(r\sin\theta\cos\theta - u_0)\csc^2\theta\}}{(r^2s^2 + 2rsu_0\cot\theta + u_0^2\csc^2\theta)^{5/2}} \tag{2.3}$$

Hence, the curvature is monotone increasing if $0 < u_0 \leq (r/2)\sin 2\theta$ and monotone decreasing if $u_0 \geq r\tan\theta$. Note with $z = \tan\theta$,

$$(\xi, \eta)\left(= \frac{z(1)}{r}\right) = \left(\frac{u_0}{2r}\left(1 + \frac{u_0}{rz}\right), \frac{1}{2}\left(\frac{u_0}{r}\right)^2\right) \tag{2.4}$$

Since $u_0 = r\sqrt{2\eta}$ and $z = \sqrt{2}\eta/(\sqrt{2}\xi - \sqrt{\eta})$, we obtain the necessary and sufficient condition for the existence of a unique quadratic segment for given (ξ, η):

Lemma 2.1 *The system of equations (2.4) has a unique solution (u_0, z) satisfying $u_0, z > 0$ if $\sqrt{2}\xi > \sqrt{\eta}$.*

Now we derive the condition for the curvature to be monotone increasing or decreasing.
Case(i) (increasing curvature): Note

$$u_0 - \frac{r\sin 2\theta}{2} = \frac{2r\sqrt{\eta}\left\{\sqrt{2}\xi^2 - 3\xi\sqrt{\eta} + \sqrt{2}\eta(1+\eta)\right\}}{2(\xi^2 - \sqrt{2}\xi\eta + \eta^2)} \tag{2.5}$$

Hence, the unique quadratic spiral with a curvature of increasing magnitude exists if

$$\xi, \eta > 0, \sqrt{2}\xi^2 - 3\xi\sqrt{\eta} + \sqrt{2}\eta(1+\eta) \leq 0 \tag{2.6}$$

Case (ii) (decreasing curvature): Note

$$u_0 - r\tan\theta = (2r\sqrt{\eta}(\xi - \sqrt{2\eta})/(\sqrt{2}\xi - \sqrt{\eta}) \tag{2.7}$$

Since $\kappa(1) = r^2\sin^2\theta/u_0^3 (> 0)$, the unique quadratic spiral with a curvature of decreasing magnitude exists if

$$\xi \geq \sqrt{2\eta}\ (> 0) \tag{2.8}$$

Thus we have

Theorem 2.1 *The reachable region of increasing one is given by (2.6) (where the equality means $\kappa'(1) = 0$) and the reachable region for a quadratic spiral of decreasing curvature magnitude is given by (2.8) (where the equality means $\kappa'(0) = 0$).*

3 T-cubic spirals

The section first treats the cases (i) and (ii) for two-parameter T-cubic spirals of the form: $z'(t) = (u(t)^2 - v(t)^2, 2u(t)v(t))$ with linear $u(t), v(t)$ where the unit tangent vectors are not fixed. Require (2.1) for $0 < \theta < \pi$ to obtain

$$u(t) = u_0(1-t) + \left\{tu_0^3/(2r)\right\}\cot(\theta/2), \quad v(t) = tu_0^3/(2r) \quad (u_0 > 0) \tag{3.1}$$

Then, with $t = 1/(1+s)$

$$\kappa'(t) = \frac{128r^3(1+s)^5\left\{rs(2r\tan\frac{\theta}{2} - u_0^2)\sin\theta + u_0^2(r\sin\theta - u_0^2)\right\}}{(1-\cos\theta)\left(4r^2s^2 + 4rsu_0^2\cot\frac{\theta}{2} + u_0^4\csc^2\frac{\theta}{2}\right)^3} \tag{3.2}$$

Hence, the curvature is monotone increasing if $0 < u_0 \leq \sqrt{r\sin\theta}$ and monotone decreasing if $u_0 \geq \sqrt{2r\tan(\theta/2)}$. Easily we obtain with $z = \tan\theta/2$

$$\xi = \frac{u_0^2}{12r}\left\{4 + \frac{2u_0^2}{rz} + \frac{u_0^4}{r^2}\left(\frac{1}{z^2} - 1\right)\right\}, \quad \eta = \frac{u_0^4}{6r^2}\left(1 + \frac{u_0^2}{rz}\right) \tag{3.3}$$

Here, we note the necessary and sufficient condition for the existence of a unique T-cubic segment for given (ξ, η):

Lemma 3.1 *The system of equations (3.3) has a unique solution (u_0, z) satisfying $u_0, z > 0$ if $\sqrt{6}\xi > (2 - 3\eta)\sqrt{\eta}$.*

Proof of lemma. As in [1], a change of variables: $u_0^2/r = g$ reduces (3.3) to

$$6\xi = 2g + g^2/z + (1-z^2)g^3/(2z^2), \; 6\eta = g^2 + g^3/z \tag{3.4}$$

Eliminate z to obtain

$$\phi(g)\left(= g^6 - 3g^4 + 12\xi g^3 - 36\eta^2\right) = 0, \quad 0 < g < \sqrt{6\eta} \tag{3.5}$$

Then,

$$(1+m)^6\phi\left(\sqrt{6\eta}/(1+m)\right) = 12\eta\sqrt{\eta}\sum_{i=0}^{6} a_i m^i, \quad m > 0 \tag{3.6}$$

where

$$(a_6, a_5, a_4) = -3\sqrt{\eta}(1, 6, 15), \quad (a_3, a_2, a_1, a_0)$$
$$= 6\sqrt{6}\left(\xi - 10\lambda, 3(\xi - 3\lambda), 3(\xi - 2\lambda), \xi - 2\lambda(1 - 9\lambda^2)\right), \; \lambda = \sqrt{\eta/6}$$

Since $2\lambda(1-9\lambda^2) < 2\lambda < 3\lambda < 10\lambda$, $a_0 \le 0$ implies $a_i < 0, 1 \le i \le 6$. Combine Descartes' rule of signs and intermediate value of theorem to give the desired result if $a_0 > 0$, i.e., $\xi > 2(1-9\lambda^2)\lambda(= (2-3\eta)\sqrt{\eta/6})$.

For $\xi > (2-3\eta)\sqrt{\eta/6}$, we obtain the condition for the curvature to be monotone increasing or decreasing.

Case (i) (increasing curvature): Note

$$u_0^2 - r\sin\theta = rg(g^6 + 3g^4 - 24\eta g + 36\eta^2)/(g^6 + g^4 - 12\eta g + 36\eta^2)\ (\le 0) \tag{3.7}$$

which requires $g^6 + 3g^4 - 24\eta g + 36\eta^2 \le 0$. Since $\phi(g) = 0$, it is reduced to

$$\psi(g)(= g^4 + 6\xi g - 12\eta) \le 0 \tag{3.8}$$

Since $\xi > (2-3\eta)\sqrt{\eta/6}$, the unique positive zero c of $\psi(g)$ is less than $\sqrt{6\eta}$. The condition for $\phi(c) \ge 0$ is equivalent to the one for the equations of $\psi(g)$ and $\phi(g) - m, m \ge 0$ to have the common zero. *Mathematica* helps us reduce their Sylvester's resultant to

$$\begin{aligned} &m^4 + 36\left\{3\xi^2 + 4\eta(1+\eta)\right\}m^3 + 432\left\{9\xi^4 + 3\xi^2\eta(8+9\eta) + 2\eta^2(9+14\eta+9\eta^2)\right\} \\ &+3888\left\{12\xi^6 + 3\xi^4(3+16\xi+24\xi^2) + 4\xi^2\eta^2(9+32\eta+27\eta^2) + 16\eta^3(3+5\eta+5\eta^2\right. \\ &\left.+3\eta^3)\right\}m + 46656\eta^2 J(\xi,\eta) \end{aligned} \tag{3.9}$$

where

$$J(\xi,\eta) = 36\xi^6 + 3\xi^4(9+16\eta+36\eta^2) - 12\xi^2\eta(6+19\eta-8\eta^2-9\eta^3) + 4\eta^2(3+2\eta+3\eta^2)^2$$

Thus, Descartes' rule of signs implies that the unique T-cubic spiral with a curvature of increasing magnitude exists if

$$J(\xi,\eta) \le 0,\ \xi,\eta > 0 \tag{3.10}$$

Case (ii) (decreasing curvature): Note

$$u_0^2 - 2r\tan(\theta/2) = 3gr(2\eta - g^2)/(6\eta - g^2)\ (\ge 0) \tag{3.11}$$

which requires $0 < g \le \sqrt{2\eta}$. Since

$$\phi(0) < 0, \quad \phi(\sqrt{2\eta}) = 4\eta\sqrt{\eta}\left\{6\xi - (6-\eta)\sqrt{2\eta}\right\}$$

Lemma 3.1 requires $\phi(\sqrt{2\eta}) \ge 0$, i.e.,

$$6\xi \ge (6-\eta)\sqrt{2\eta}, \eta > 0 \tag{3.12}$$

Since $\kappa(1) = (16r^3/u_0^8)\sin^4(\theta/2)\ (> 0)$, the reachable region for the unique T-cubic spiral with a curvature of decreasing magnitude is given by (3.12). Hence we have

Theorem 3.1 *The reachable region for a T-cubic spiral of increasing curvature magnitude is given by (3.10) (where the equality means $\kappa'(1) = 0$) and the reachable region of decreasing one is given by (3.12) (where the equality means $\kappa'(0) = 0$).*

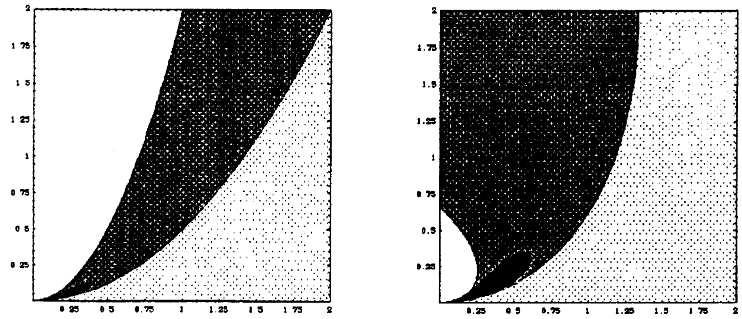

Fig. 1 (Cases (i)(heavy dots)-(ii)(light dots)). Regions for quadratic (left) and T-cubic (right) spirals.

4 General cubic spirals

This section treats cases (iii) - (iv) for general cubic two-parameter spirals where the ending tangent vector is fixed.

Case (iii) (increasing curvature): Require for fixed $0 < \theta < \pi/2$:

$$\mathbf{z}(0) = (0,0), \quad \mathbf{z}'(0) \parallel (1,0), \quad \kappa(0) = 0, \quad \mathbf{z}'(1) \parallel (\cos\theta, \sin\theta), \quad \kappa(1) = 1/r \tag{4.1}$$

to obtain

$$x(t) = \{qrt/(6\sin\theta)\}\left[q\left\{(3-2t)t + m(3-3t+t^2)\right\} + t^2\sin 2\theta\right], y(t) = qrt^3\sin\theta/3 \tag{4.2}$$

A symbolic manipulator helps us obtain

$$\left\{x'(t)^2 + y'(t)^2\right\}^{5/2}\kappa'(t) = \left[q^5r^4/\left\{4(1+s)^5\sin^2\theta\right\}\right]\sum_{i=0}^{5} b_i s^i, \ t = 1/(1+s) \tag{4.3}$$

where

$$b_0 = 4\{3q\cos\theta - (4+m)\sin\theta\}\sin\theta, b_1 = 2\left\{6q^2 - q(5-4m)\sin 2\theta - 10m\sin^2\theta\right\}$$
$$b_2 = 2q\{(-2+13m)q - 2m(4-m)\sin 2\theta\}, b_3 = 2mq\{(-3+10m)q - 2m\sin 2\theta\}$$
$$b_4 = 5m^3q^2, b_5 = m^3q^2$$

Hence, we obtain a sufficient spiral condition, i.e., $b_i \geq 0, 0 \leq i \leq 5$:

Lemma 4.1 *The general cubic segment $z(t), 0 \leq t \leq 1$ of the form (4.2) is a spiral satisfying (4.1) if $m > 3/10$ and*

$$q \geq q(m,\theta) \left(= Max\left[\frac{(4+m)\tan\theta}{3}, \frac{2m(4-m)\sin 2\theta}{13m-2}, \frac{2m\sin 2\theta}{10m-3},\right.\right.$$
$$\left.\left.\frac{1}{6}\left\{(5-4m)\cos\theta + \sqrt{60m + (5-4m)^2\cos^2\theta}\right\}\sin\theta\right]\right) \tag{4.4}$$

where $q(m,\theta) = \{(4+m)/3\}\tan\theta$ *for* $m \geq 2(\sqrt{6}-1)/5 (\approx 0.5797)$.

From (4.2), we have

$$(\xi,\eta) = \left(q\left\{(1+m)q + \sin 2\theta\right\}/(6\sin\theta), q\sin\theta/3\right) \tag{4.5}$$

Solve (4.5) for m, q to obtain

$$(m,q) = \Big((-3\eta^2 + 2\xi\sin^3\theta - \eta\sin\theta\sin 2\theta)/(3\eta^2), 3\eta/(\sin\theta)\Big) \tag{4.6}$$

Note $q \geq \{(4+m)/3\}\tan\theta$ and $m \geq 2(\sqrt{6}-1)/5$ to obtain the reachable region (indicated by heavy dots in Fig. 2 (left)) for the end points of the general cubic spiral where

$$\frac{3(3+2\sqrt{6})\eta^2 + 5\eta\sin\theta\sin 2\theta}{10\sin^3\theta} \leq \xi \leq \frac{27\eta^3 - 9\eta^2\sin\theta\tan\theta + 2\eta\sin^4\theta}{2\sin^4\tan\theta} \tag{4.7}$$

In addition,

$$\frac{\eta}{\xi} \leq \frac{3\sin 2\theta}{(1+m)(4+m) + 6\cos^2\theta} \tag{4.8}$$

Note that $m = 1$ and $\kappa'(1) = 0$ (i.e., $q = (4+m)/3\tan\theta$) are fixed in Walton & Meek([1]) where the reachable region reduces to a single point:

$$\xi = 5\left(3\sin\theta + 5\sec\theta\tan\theta\right)/27, \quad \eta = 5\sin\theta\tan\theta/9 \tag{4.9}$$

Case (iv) (decreasing curvature): Require for fixed $0 < \theta < \pi/2$:

$$z(0) = (0,0), \quad z'(0) \parallel (1,0), \quad \kappa(0) = 1/r, \quad z'(1) \parallel (\cos\theta, \sin\theta), \quad \kappa(1) = 0 \tag{4.10}$$

Then, transformation, i.e., rotation, shift, reflection with respect to y-axis and change of variable t with $1-t$ to (4.2) gives

$$\begin{aligned} x(t) &= \{qrt/(6\sin\theta)\}\left[qt\{3-(2-m)t\}\cos\theta + 2(3-3t+t^2)\sin\theta\right] \\ y(t) &= (q^2rt^2/6)\{3-(2-m)t\} \end{aligned} \tag{4.11}$$

Note that Lemma 4.1 is valid under the above transformation, or directly

$$\left\{x'(t)^2 + y'(t)^2\right\}^{5/2}\kappa'(t) = -\left[q^5r^4/\left\{4(1+s)^5\sin^2\theta\right\}\right]\sum_{i=0}^{5} b_i s^{5-i}, \ t = 1/(1+s) \tag{4.12}$$

Note

$$(\xi,\eta) = \Big(q\left\{(1+m)q\cos\theta + 2\sin\theta\right\}/(6\sin\theta), (1+m)q^2/6\Big) \tag{4.13}$$

Solve (4.13) for m, q to obtain

$$(m,q) = \left(\frac{2\eta - 3\xi^2 + 6\xi\eta\cot\theta - 3\eta^2\cot^2\theta}{3(\xi-\eta\cot\theta)^2}, 3(\xi - \eta\cot\theta)\right) \tag{4.14}$$

As in the above Case (iii), we obtain the reachable region (indicated by light dots in Fig. 2 (left)) for the end points of the general cubic spiral

$$
\begin{aligned}
&27(\xi - \eta\cot\theta)^3 \geq 9\tan\theta(\xi - \eta\cot\theta)^2 + 2\eta\tan\theta \\
&10\eta - 3(3 + 2\sqrt{6})(\xi - \eta\cot\theta)^2 \geq 0
\end{aligned}
\tag{4.15}
$$

Note that $m = 1$ and $\kappa'(0) = 0$ (i.e., $q = (4+m)/3\tan\theta$) are fixed in Walton & Meek([1]) where the reachable region reduces to a single point:

$$\xi = 40\tan\theta/27, \quad \eta = 25\tan^2\theta/27 \tag{4.16}$$

5 PH quintic spirals

This section treats cases (iii) and (iv) for two-parameter PH-quintic spiral segment of the form: $\boldsymbol{z}'(t) = (u(t)^2 - v(t)^2, 2u(t)v(t))$. For later use, we note

$$
\begin{aligned}
&\left\{u^2(t) + v^2(t)\right\}^3 \kappa'(t) = 2\left[\{u(t)v''(t) - u''(t)v(t)\}\left\{u^2(t) + v^2(t)\right\}\right. \\
&\left.-4\{u(t)v'(t) - u'(t)v(t)\}\{u(t)u'(t) + v(t)v'(t)\}\right] (= w(t))
\end{aligned}
\tag{5.1}
$$

Case (iii) (increasing curvature): Require (4.1) for fixed $0 < \theta < \pi$ to obtain

$$\frac{u(t)}{\sqrt{r}} = \frac{\sqrt{q}}{4\sin\frac{\theta}{2}}\left[q\{m(1-t) + 2t\}(1-t) + 2t^2\sin\theta\right], \quad \frac{v(t)}{\sqrt{r}} = \sqrt{q}t^2\sin\frac{\theta}{2} \tag{5.2}$$

A symbolic manipulator helps us obtain

$$w(t) = \left[q^3r^2/\left\{16(1+s)^5\sin^2\frac{\theta}{2}\right\}\right]\sum_{i=0}^{5} c_i s^i, \; t = 1/(1+s) \tag{5.3}$$

where

$$
\begin{aligned}
&c_0 = 16\left\{q\sin\theta - (6+m)\sin^2\frac{\theta}{2}\right\}, c_1 = 8\left\{2q^2 - q(4-3m)\sin\theta - 14m\sin^2\frac{\theta}{2}\right\} \\
&c_2 = 4q\{(-2+9m)q - 3(4-m)\sin\theta\}, c_3 = 4mq\{(-3+7m)q - 3m\sin\theta\} \\
&c_4 = m^2q^2(-2+7m), c_5 = m^3q^2
\end{aligned}
$$

Hence, we obtain a sufficient spiral condition $c_i, 0 \leq i \leq 5$ for $\boldsymbol{z}(t)$:

Lemma 5.1 *The PH-quintic segment $z(t), 0 \leq t \leq 1$ of the form (5.3) is a spiral satisfying (4.1) if $m > 3/7$ and*

$$
\begin{aligned}
&q \geq q(m,\theta) \left(= Max\left[\frac{6+m}{2}\tan\frac{\theta}{2}, \frac{3m(4-m)\sin\theta}{9m-2}, \frac{3m\sin\theta}{7m-3},\right.\right. \\
&\left.\left.\frac{1}{4}\left\{(4-3m)\sin\theta + \sqrt{56m(1-\cos\theta) + (4-3m)^2\sin^2\theta}\right\}\right]\right)
\end{aligned}
\tag{5.4}
$$

where $q(m,\theta) = \{(6+m)/2\}\tan(\theta/2)$ for $m \geq 2(-3+\sqrt{30})/7 (\approx 0.707)$.

With help of *Mathematica* (if necessary),

$$(i) \quad \xi = \frac{q\left\{(2+3m+3m^2)q^2 + 2q(3+m)\sin\theta + 24\cos\theta(1-\cos\theta)\right\}}{120(1-\cos\theta)}$$

$$(ii) \quad \eta = \frac{q\left\{(3+m)q + 12\sin\theta\right\}}{60} \tag{5.5}$$

Unlike in general cubic spirals, it is not easy to solve for m, q and so the reachable region (indicated by heavy dots in Fig. 2 (right)) is numerically determined. In addition,

$$\frac{\eta}{\xi} \le \frac{4(42+9m+m^2+24\cos\theta)\sin\theta}{4(42+9m+m^2)\cos\theta + 3(48+56m+50m^2+13m^3+m^4+32\cos^2\theta)} \tag{5.6}$$

Note that $m = 1$ and $\kappa'(1) = 0$ (i.e., $q = \{(6+m)/2\}\tan(\theta/2)$) are fixed in [1] where the reachable region reduces to a single point:

$$\xi = 7\,(69 + 26\cos\theta + 6\cos 2\theta)\sec^2(\theta/2)\tan(\theta/2)/240$$

$$\eta = 7(13 + 6\cos\theta)\tan^2(\theta/2)/60 \tag{5.7}$$

Case (iv) (decreasing curvature): Require (4.8) for fixed $0 < \theta < \pi$. Then, transformation, i.e., rotation, shift, reflection with respect to y-axis and change of variable t with $1-t$ to (5.2) gives

$$\frac{u(t)}{\sqrt{r}} = \frac{\sqrt{q}}{4\sin\frac{\theta}{2}}\left[qt\left\{2-(2-m)t\right\}\cos\frac{\theta}{2} + 4(1-t)^2\sin\frac{\theta}{2}\right], \frac{v(t)}{\sqrt{r}} = \frac{qt\sqrt{q}}{4}\left\{2-(2-m)t\right\} \tag{5.8}$$

Then, note that Lemma 5.1 remains valid under the transformation, i.e., rotation, shift, reflection and change of variable, or directly

$$w(t) = -\left[q^3r^2/\left\{16(1+s)^5\sin^2(\theta/2)\right\}\right]\sum_{i=0}^{5} c_i s^{5-i}, \; t = 1/(1+s) \tag{5.9}$$

With help of *Mathematica* (if necessary),

$$(i) \quad \xi = \frac{q\left[24 + \left\{-24 + (2+3m+3m^2)q^2\right\}\cos\theta + 2q(3+m)\sin\theta\right]}{120(1-\cos\theta)}$$

$$(ii) \quad \eta = \frac{q^2\left\{2(3+m)\sin\theta + q(2+3m+3m^2)(1+\cos\theta)\right\}}{120\sin\theta} \tag{5.10}$$

The heavy and light dotted regions correspond to the cases (iii) and (iv), respectively. Note that $m = 1$ and $\kappa'(0) = 0$ (i.e., $q = \{(6+m)/2\}\tan(\theta/2)$) are fixed in Walton & Meek ([1]) where the region reduces to a single point:

$$\xi = 7(26 + 75\cos\theta)\sec^2(\theta/2)\tan(\theta/2)/240, \quad \eta = 147\tan^2(\theta/2)/40 \tag{5.11}$$

Walton & Meek's points by (4.9), (4.16) for general cubics and (5.7), (5.11) for PH-quintic are denoted in black discs.

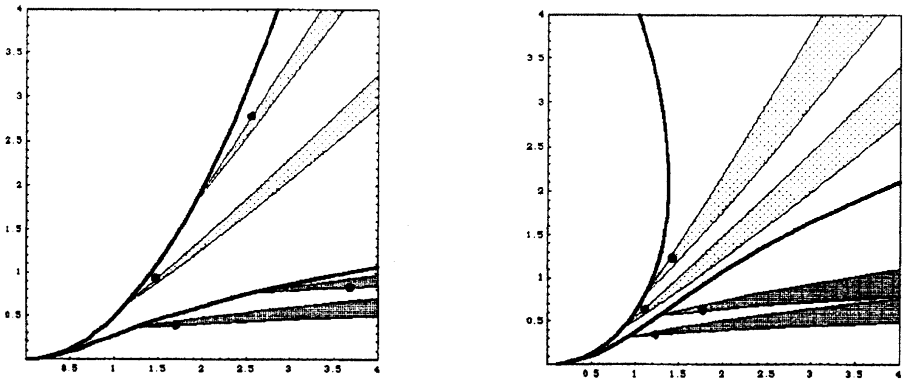

Fig. 2 (Cases (iii)(light dotted region)-(iv)(heavy dotted region)). Regions for general cubic (left) and PH-quintic (right) spirals for $\theta = \pi/4(lower), \pi/3(upper)$.

6 Numerical Examples

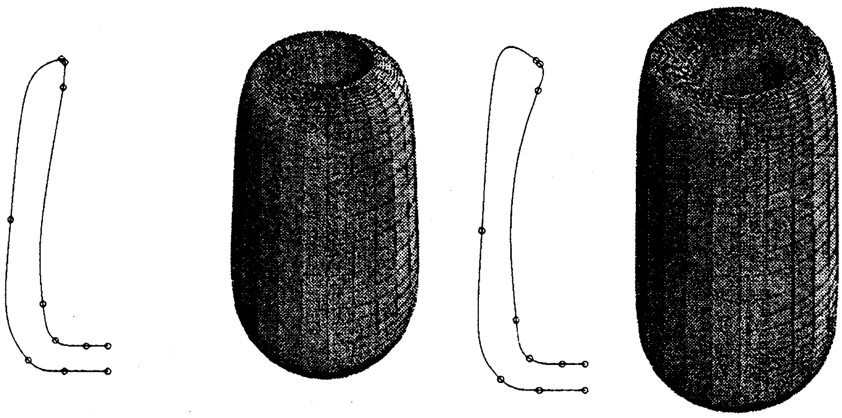

Fig. 3. Vase profiles with G^2 cubic Bézier spiral segments and their shaded renditions.

References

[1] D. Walton and D. Meek(1998), Planar G^2 curve design with spiral segments, Computer-Aided Design, 30, 529-538, 72, 85-100.

[2] D. Walton and D. Meek(1999), Planar G^2 transition between two circles with a fair cubic Bézier curve, Computer-Aided Design 31, 857-866.

[3] D. Walton and D. Meek(2002), Planar G^2 transition with a fair Pythagorean hodograph quintic curve, J. Comput. Appl. Math., 138, 109-126.